2020 B

Week 7 (March 3)

Recall the following steps in applying the change of variables formula

(I) Decide $u = g(x,y)$, $v = h(x,y)$. Sketch G

(II) Solve for $x = j(u,v)$, $y = k(u,v)$. Calculate $\frac{\partial(x,y)}{\partial(u,v)}$.

(III) Use $\displaystyle\iint_D f = \iint_G \hat{f} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v)$

e.g. Find the area of the region bdd by

$xy = 1/2$, $xy = 2$, $y = x/2$, $y = x$.

(I) Let $u = xy$, $v = y/x$. Then $G: \frac{1}{2} \le u \le 2$

$\frac{1}{2} \le v \le 2$.

(II) $uv = xy\,\frac{y}{x} = y^2 \Rightarrow y = \sqrt{uv}$

$\Rightarrow x = u/y = \sqrt{u}/\sqrt{v}$.

$\therefore$ $j(u,v) = u^{\frac{1}{2}} v^{-\frac{1}{2}}$

$k(u,v) = u^{\frac{1}{2}} v^{\frac{1}{2}}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}} & -\frac{1}{2} u^{\frac{1}{2}} v^{-3/2} \\ \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} & \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}} \end{vmatrix} = \frac{1}{2}\frac{1}{v}.$$

(III) $\displaystyle\iint_D 1\, dA = \int_{1/2}^{2}\int_{1/2}^{2} 1\, \frac{1}{2v}\, du\, dv$

$\vdots$

$= \frac{3}{2} \ln 2$.

We will show that

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|},$$

and this simplifies Step II. Let's look at the example.

$$u = xy, \quad v = y/x$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = 2\frac{y}{x} = 2v$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = 2v = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

So. $\iint_D 1\,dA = \int_{1/2}^{2}\int_{1/2}^{2} \frac{1}{2v}\,du\,dv = \cdots = \frac{3}{2}\ln 2$ as before.

<u>Theorem</u> Let $\Phi = (g,h)$ be 1-1 onto from G to D and its inverse Φ^{-1}. Then

$$I = \nabla\Phi^{-1}\cdot\nabla\Phi$$ where I is the identity matrix, $\nabla\Phi$ is the jacobian matrix of Φ.

Pf. Let $\Phi(u,v) = (g(u,v), h(u,v))$
$\Phi^{-1}(x,y) = (j(x,y), k(x,y))$.

We have

$$\begin{aligned}(u,v) &= \Phi^{-1}\circ\Phi(u,v)\\ &= \Phi^{-1}(g(u,v), h(u,v))\\ &= (j(g(u,v),h(u,v)),\ k(g(u,v),h(u,v)))\end{aligned}$$

or

$$u = j(g(u,v), h(u,v))$$
$$v = k(g(u,v), h(u,v))$$

Diff. and use Chain rule,

$$\frac{\partial u}{\partial u} = 1 = j_x g_u + j_y h_u$$

$$\frac{\partial u}{\partial v} = 0 = j_x g_v + j_y h_v$$

$$\frac{\partial v}{\partial u} = 0 = k_x g_u + k_y h_u$$

$$\frac{\partial v}{\partial v} = 1 = k_x g_v + k_y h_v$$

These 4 equations can be put into

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} j_x & j_y \\ k_x & k_y \end{pmatrix} \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}, \text{ ie}$$

$$I = (\nabla \Phi^{-1})(\nabla \Phi).$$

<u>Corollary</u> $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|}$.

Pf: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} j_x & j_y \\ k_x & k_y \end{pmatrix} \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}$

$$1 = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det \begin{pmatrix} j_x & j_y \\ k_x & k_y \end{pmatrix} \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}$$

$$= \det \begin{pmatrix} j_x & j_y \\ k_x & k_y \end{pmatrix} \det \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}$$

$$= \det \nabla \Phi^{-1} \cdot \det \nabla \Phi.$$

So
$$1 = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)}.$$

Taking absolute value, we get it.

e.g. (cont'd) use the previous D and G but now consider

$$\iint_D y^2 \, dA(x,y)$$

Since $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2v}$,

$$\iint_D y^2 \, dA(x,y) = \int_{1/2}^{2}\int_{1/2}^{2} y^2 \frac{1}{2v} \, du\, dv.$$

Here, we still need to express y in terms of (u,v), but that is easy: $y^2 = uv$

$$\therefore \quad = \int_{1/2}^{2}\int_{1/2}^{2} uv \frac{1}{2v} \, du\, dv$$

$$= \int_{1/2}^{2}\int_{1/2}^{2} \frac{u}{2} \, du\, dv = \cdots \quad \#$$

Step II cannot be skipped completely.

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For $\Phi : G \to \Omega$ 1-1 onto

$$\Phi(u,v,w) = (\,g(u,v,w),\ h(u,v,w),\ k(u,v,w))$$

the formula holds :

$$\iiint_{\Omega} f = \iiint_{G} \hat{f}(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dA(u,v,w) .$$

It suffices to look at some examples.

e.g (spherical coordinates)

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi .$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} x_\rho & x_\varphi & x_\theta \\ y_\rho & y_\varphi & y_\theta \\ z_\rho & z_\varphi & z_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \sin\varphi \cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \sin\varphi\cos\theta\,(\rho^2\sin^2\varphi\cos\theta) - \rho\cos\varphi\cos\theta\,(-\rho\sin\varphi\cos\theta\cos\varphi)$$
$$+ -\rho\sin\varphi\sin\theta\,(-\rho\sin^2\varphi\sin\theta - \rho\cos^2\varphi\sin\theta)$$

$$= \rho^2\sin^3\varphi\cos^2\theta + \rho^2\cos^2\varphi\sin\varphi\cos^2\theta$$
$$+ \rho^2\sin^3\varphi\sin^2\theta + \rho^2\sin\varphi\cos^2\varphi\sin^2\theta$$

$$= \rho^2\sin^3\varphi + \rho^2\sin\varphi\cos^2\varphi$$

$$= \rho^2\sin\varphi .$$